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Entanglement in the scattering process by the spin impurity

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Abstract
We study entanglement in the scattering processes by the spin-fixed impurity and the Kondo impurity, respectively. The spin-fixed impurity is employed as a filter to concentrate entanglement. One Kondo impurity can create entanglement between two noninteracting particles and one scattered particle can do so between two noninteracting Kondo impurities.

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1. Introduction

Entanglement lies at the heart of quantum information and quantum computation [1], which is responsible for many quantum protocols, such as quantum teleportation [2], dense coding [3], quantum cryptography [4, 5]. Now entanglement is regarded as a valuable resource in quantum information processing. Much effort was devoted to exploring the entanglement creation process since entanglement cannot be created from scratch between two noninteracting systems. For example, Childs *et al* [6] found an explicit formula for the maximum entanglement created by a class of two-qubit interaction Hamiltonians, including the Ising interaction and the anisotropic Heisenberg interaction. Nielsen *et al* [7] developed a theory quantifying the strength of quantum dynamical operations. Maximally entangled state is required in most of the quantum protocols. Therefore, entanglement concentration is a basic process [8] in which a filter [9] plays an important role. Experimentally these tasks are mainly realized in quantum optics. In this paper, we study entanglement concentration and entanglement creation during the scattering processes by the δ interaction produced by the spin-fixed impurity and Kondo impurity, respectively. Specifically, the paper is structured as follows. Section 2 shows how the spin-fixed impurity plays a role as a filter during entanglement concentration. Section 3 shows how one Kondo impurity makes two noninteracting particles entangled, and how a moving particle makes two noninteracting Kondo impurities entangled. Section 4 concludes with a brief summary.

2. Entanglement concentration

We consider the scattering process where a free particle with spin-$\frac{1}{2}$ is scattered by a spin-fixed impurity and show that the impurity can be used as a spin-state filter to concentrate entanglement. Before that, let us recall the problem of the free particle scattered by the δ potential in many standard Quantum Mechanics books. The Hamiltonian is

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r\delta(x), \tag{1}$$

where we set $\hbar = m = 1$ and r is the strength of the δ potential. The δ potential comes from a local impurity without a spin. Suppose the incident wavefunction is $\phi_i = \mathrm{e}^{\mathrm{i}kx}$, then the solution of the scattering process is

$$\phi_f = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R\,\mathrm{e}^{-\mathrm{i}kx}, & x < 0, \\ S\,\mathrm{e}^{\mathrm{i}kx}, & x > 0, \end{cases} \tag{2}$$

where $S = 1/(1 + \mathrm{i}\xi)$ is the transmission amplitude, $R = S - 1$ the reflection amplitude, $\xi = r/k$.

Now we consider the scattering process in which a free particle with spin-$\frac{1}{2}$ is scattered by an impurity with a spin-fixed state. The situation occurs where a free electron is scattered by a localized electron whose spin is fixed, saying spin-up. If the electric interaction is neglected, the scattering process can be described by the Toy model

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r\delta(x)(1 - \sigma_z), \tag{3}$$

where σ_z is the Pauli operator. In the eigenbasis of σ_z, the potential is written as $2r\delta(x)\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. That is to say the spin of the particle is visible to the impurity. If the particle is spin-up, the impurity lets the particle go through freely, just as the impurity is nonexisting. If the particle is spin-down, the impurity acts as the $2r\delta(x)$ potential. For the general case where the particle is in the coherent state of spin-up and spin-down, the problem can be solved in the spirit of the partial-wave method. The matrix in the potential term is diagonalized and the eigenvalues and eigenstates are found; for each of the eigenstates, the scattering problem is solved for the potential determined by its corresponding eigenvalue; the incident state is expanded in the eigenstates and the scattering state is the coherent superposition of the scattering states of the eigenstates. Now suppose the incident state is the following form:

$$\phi_i = \mathrm{e}^{\mathrm{i}kx}(\alpha|0\rangle + \beta|1\rangle), \tag{4}$$

where $|0\rangle$ is spin-up and $|1\rangle$ spin-down, and $|\alpha|^2 + |\beta|^2 = 1$. The solution can be obtained,

$$\phi_f = \begin{cases} \alpha\,\mathrm{e}^{\mathrm{i}kx}|0\rangle + \beta(\mathrm{e}^{\mathrm{i}kx} + R\,\mathrm{e}^{-\mathrm{i}kx})|1\rangle, & x < 0, \\ \mathrm{e}^{\mathrm{i}kx}(\alpha|0\rangle + \beta S|1\rangle), & x > 0, \end{cases} \tag{5}$$

where $S = 1/(1 + 2\mathrm{i}\xi)$, $\xi = r/k$. Note that in the transmitting part $x > 0$, the amplitude of the spin-down is decreased. The effect of the scattering process on the spin state is like a spin filter. Strictly speaking, the spin-$\frac{1}{2}$ filter is called as a qubit filter, which is a kind of two-outcome positive operator-valued measurement (POVM), where it changes the ratio of the amplitudes in the coherent state.

A spin filter can be employed to concentrate entanglement between two non-maximally entangled particles [9]. Suppose the state of two particles is

$$\phi_i = \mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{e}^{\mathrm{i}kx_1}(a|00\rangle + b|11\rangle), \qquad |a| < |b|, \tag{6}$$

in which $|a|^2 + |b|^2 = 1$. The entanglement of the pure bipartite state is measured by von Neumann entropy of the reduced state of any particle, $E = -\mathrm{tr}\,\rho_1 \log \rho_1 = -|a|^2 \log |a|^2 - |b|^2 \log |b|^2$, where $\rho_1 = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|$. Let particle 1 be scattered by the impurity. According to the partial-wave method,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}kx_1}|0\rangle &\rightarrow \begin{cases} \mathrm{e}^{\mathrm{i}kx_1}|0\rangle, & x < 0, \\ \mathrm{e}^{\mathrm{i}kx_1}|0\rangle, & x > 0, \end{cases} \\
\mathrm{e}^{\mathrm{i}kx_1}|1\rangle &\rightarrow \begin{cases} (\mathrm{e}^{\mathrm{i}kx_1} + R\,\mathrm{e}^{-\mathrm{i}kx_1})|1\rangle, & x < 0, \\ S\,\mathrm{e}^{\mathrm{i}kx_1}|1\rangle, & x > 0. \end{cases}
\end{aligned} \tag{7}$$

After the scattering process, the scattering state of the two particles is of the form

$$\phi_f = \begin{cases} a\,\mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{e}^{\mathrm{i}kx_1}|00\rangle + b\,\mathrm{e}^{-\mathrm{i}kx_2}(\mathrm{e}^{\mathrm{i}kx_1} + R\,\mathrm{e}^{-\mathrm{i}kx_1})|11\rangle, & x < 0, \\ \mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{e}^{\mathrm{i}kx_1}(a|00\rangle + bS|11\rangle), & x > 0. \end{cases} \tag{8}$$

Projecting the state on the reflecting part ($x < 0$) and the transmitting one ($x > 0$) means performing measurement on the position of the particle. We concern mainly about the spin state in the event when we observe the particle appears in $x > 0$. If $|a| = |bS|$, the two particles are maximally entangled. Since S is dependent on r/k, for a given a, we can adjust k or r such that $|a| = |bS|$ is satisfied. Note that the probability to obtain a maximally entangled state is less than the maximum probability $2|a|^2$ that can be obtained theoretically [10]. The reason lies at the reflecting state still contains entanglement. Possibly, there exists the optimal direction of the impurity to maximize the probability.

3. Entanglement creation

We discuss the scattering process where a free particle is scattered by the Kondo impurity, then study entanglement creation processes between two noninteracting particles and between two noninteracting Kondo impurities, respectively.

The impurity is called Kondo impurity if the spin of the impurity is not fixed and interacts with the particle. The scattering process is described by the Kondo model,

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r\delta(x)\vec{\sigma}_1 \cdot \vec{\sigma}_0, \tag{9}$$

where $\vec{\sigma}_1$ is the spin operator vector of the particle and $\vec{\sigma}_0$ that of the Kondo impurity. The eigenbasis of $\vec{\sigma}_1 \cdot \vec{\sigma}_0$ is

$$\begin{aligned}
|\lambda_1\rangle &= |00\rangle, \\
|\lambda_2\rangle &= |11\rangle, \\
|\lambda_3\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\lambda_4\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle),
\end{aligned} \tag{10}$$

where λ_i are the corresponding eigenvalues, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$, $\lambda_4 = 0$. In the eigenbasis, H is expressed as

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r\delta(x)\sum_i \lambda_i|\lambda_i\rangle\langle\lambda_i|. \tag{11}$$

When the incident state is $\phi_i = \mathrm{e}^{\mathrm{i}kx}|\lambda_i\rangle$, the solution of the scattering problem gives

$$\phi_f = \begin{cases} (\mathrm{e}^{\mathrm{i}kx} + R_i\,\mathrm{e}^{-\mathrm{i}kx})|\lambda_i\rangle, & x < 0, \\ S_i\,\mathrm{e}^{\mathrm{i}kx}|\lambda_i\rangle, & x > 0, \end{cases} \tag{12}$$

where $S_i = 1/(1 + \mathrm{i}\lambda_i\xi)$, $\xi = r/k$.

The generic incident state is

$$\phi_i = \mathrm{e}^{\mathrm{i}kx}|\chi\rangle = \mathrm{e}^{\mathrm{i}kx}\sum_i c_i|\lambda_i\rangle, \tag{13}$$

and the scattering state is

$$\phi_f = \begin{cases} \sum_i c_i(\mathrm{e}^{\mathrm{i}kx} + R_i\,\mathrm{e}^{-\mathrm{i}kx})|\lambda_i\rangle, & x < 0, \\ \mathrm{e}^{\mathrm{i}kx}\sum_i c_i S_i|\lambda_i\rangle, & x > 0. \end{cases} \tag{14}$$

After the scattering process ϕ_f is projected onto two subspaces. One is $x < 0$ and the other is $x > 0$. The transmitting part $x > 0$ is retained while the reflecting one $x < 0$ discarded. In other words, we only concern about the transmitting state. Because we pay attention to entanglement between the spins and the wavefunction of coordinate in the transmitting state is the same as that in the incident one, we omit the wavefunction of coordinate and rewrite the evolution of the spins,

$$\begin{aligned} |00\rangle &\to S_1|00\rangle, \\ |11\rangle &\to S_2|11\rangle, \\ |01\rangle &\to \frac{S_3 + S_4}{2}|01\rangle + \frac{S_3 - S_4}{2}|10\rangle, \\ |10\rangle &\to \frac{S_3 - S_4}{2}|01\rangle - \frac{S_3 + S_4}{2}|10\rangle. \end{aligned} \tag{15}$$

Suppose, initially, the impurity is polarized in spin-up and the particle is in a generic state, then

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \to \alpha S_1|00\rangle + \frac{\beta(S_3 - S_4)}{2}|01\rangle - \frac{\beta(S_3 + S_4)}{2}|10\rangle. \tag{16}$$

Once the particle goes through the Kondo impurity, the spin of the particle is entangled with that of the impurity.

The Kondo impurity can be employed to create entanglement between two noninteracting scattering particles. The process is as follows. Suppose initially the Kondo impurity K is polarized in $|1\rangle_K$, the two noninteracting particles P_B and P_A in the product state $|0\rangle_B|0\rangle_A$. (1) Let P_A fly to K and be scattered by K. (2a) If P_A is reflected, then the impurity K is polarized again in $|1\rangle_K$ and a new P_A is chosen to go to (1); (2b) If P_A passes, then P_B is prepared to fly to K. (3a) If P_B comes back, then (1) is restarted; (3b) If P_B goes through, we obtain the desired case in which both P_A and P_B successfully transmit through the impurity K. (4) The measurement is performed on the impurity K in the basis $\{|0\rangle_K, |1\rangle_K\}$. (5a) If the outcome is $|1\rangle_K$, we begin (1) again; (5b) Once the outcome $|0\rangle_K$ happens, entanglement between P_A and P_B is created. The selective evolution (1) $\to$ (2b) $\to$ (3b) $\to$ (4) $\to$ (5b) is expressed as

$$\begin{aligned} |0\rangle_B|0\rangle_A|1\rangle_K &\to |0\rangle_B\left(\frac{S_3^A + S_4^A}{2}|0\rangle_A|1\rangle_K + \frac{S_3^A - S_4^A}{2}|1\rangle_A|0\rangle_K\right) \\ &\to \frac{S_3^B + S_4^B}{2}\frac{S_3^A + S_4^A}{2}|0\rangle_B|0\rangle_A|1\rangle_K + \left(\frac{S_3^B - S_4^B}{2}\frac{S_3^A + S_4^A}{2}|1\rangle_B|0\rangle_A\right. \end{aligned}$$

$$+ \frac{S_1^B\left(S_3^A - S_4^A\right)}{2}|0\rangle_B|1\rangle_A\Bigg)|0\rangle_K,$$

$$\rightarrow \left(\frac{S_3^B - S_4^B}{2}\frac{S_3^A + S_4^A}{2}|1\rangle_B|0\rangle_A + \frac{S_1^B\left(S_3^A - S_4^A\right)}{2}|0\rangle_B|1\rangle_A\right)|0\rangle_K. \tag{17}$$

Here, we remark that it is important to prepare the initial spin state of the particles and the Kondo impurity properly. Explicitly, when the initial state of the three particles is $|0\rangle_B|0\rangle_A|0\rangle_K$, no entanglement will exist in the final transmitting state. Note that $S_i^{A(B)} = 1/(1 + \mathrm{i}\lambda_i\xi_{A(B)})$, where $\xi_{A(B)} = r/k_{A(B)}$ can be modulated. So it is even possible to obtain maximally entangled state directly between P_A and P_B if the condition $\left|\frac{S_3^B - S_4^B}{2}\frac{S_3^A + S_4^A}{2}\right| = \left|\frac{S_1^B(S_3^A - S_4^A)}{2}\right|$ is satisfied. The equation for obtaining maximally entangled state is

$$\xi_A^2 = \frac{\xi_B^2\left(1 + \xi_B^2\right)}{1 + 3\xi_B^2 - \xi_B^4}. \tag{18}$$

Exchanging the roles of the scattering particle and the impurity, a moving particle can create entanglement between two separate noninteracting impurities. Suppose K_A, K_B are two Kondo impurities located at $-a$ and a, respectively. A particle P goes from left to right scattering with K_A and K_B successively. The evolution is described by the Hamiltonian

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r_A\delta(x + a)\vec{\sigma}\cdot\vec{\sigma_A} + r_B\delta(x - a)\vec{\sigma}\cdot\vec{\sigma_B}. \tag{19}$$

Strictly speaking, we should solve the multiple scattering problem. For simplicity, we only consider that P goes through K_A and K_B directly where it contributes mostly to the transmitting state and the high-order contribution from the reflection to and fro between the impurities is neglected. Entanglement creation between K_A and K_B is similar as follows:

$$\begin{aligned}
|1\rangle_P|0\rangle_A|0\rangle_B &\rightarrow \left(\frac{S_3^A - S_4^A}{2}|0\rangle_P|1\rangle_A - \frac{S_3^A + S_4^A}{2}|1\rangle_P|0\rangle_A\right)|0\rangle_B \\
&\rightarrow \frac{S_3^A + S_4^A}{2}\frac{S_3^B + S_4^B}{2}|1\rangle_P|0\rangle_A|0\rangle_B + |0\rangle_P\left(\frac{S_1^B\left(S_3^A - S_4^A\right)}{2}|1\rangle_A|0\rangle_B\right. \\
&\quad \left. - \frac{S_3^A + S_4^A}{2}\frac{S_3^B - S_4^B}{2}|0\rangle_A|1\rangle_B\right) \\
&\rightarrow |0\rangle_P\left(\frac{S_1^B\left(S_3^A - S_4^A\right)}{2}|1\rangle_A|0\rangle_B - \frac{S_3^A + S_4^A}{2}\frac{S_3^B - S_4^B}{2}|0\rangle_A|1\rangle_B\right).
\end{aligned} \tag{20}$$

4. Summary

In this paper, we showed the scattering process can be employed to manipulate entanglement. When one of two non-maximally entangled particles is scattered by a spin-fixed impurity, the impurity, acting as a filter, can make the two particles maximally entangled that realizes entanglement concentration. Transmitting through the same Kondo impurity sequently, two noninteracting particles become entangled if the proper outcome of the measurement on the Kondo impurity occurs. Similarly, successfully passed by the same flying particle, two noninteracting Kondo impurities get entangled if the proper outcome of the measurement on the particle happens. These processes realize entanglement creation. In the process of entanglement manipulation, we consider how we obtain maximally entangled state. Actually, maximally entangled state is produced probabilistically. If we choose the proper input state and the proper measurement, the probability of the occurrence may be optimized. Also we just consider the scattering process with two Kondo impurities in the first order. Strictly speaking, the scattering process with high orders should be included and deserves further investigation.

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